Stat 444
Advanced Long-term Actuarial Math

Lecture 3: Multiple Life and Multiple Decrement Models

Multiple Life Models

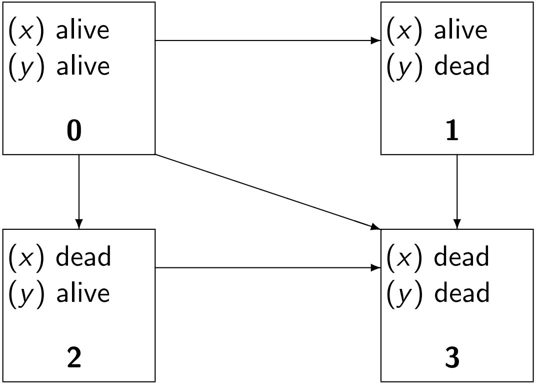
To this point, all of the products we've dealt with were based on the life status (alive / dead or multi-state) of a single individual.

However, in order to deal with products involving multiple lives, we need more complex models.

- These future lifetimes may be dependent or independent.

The force of transition from state i to state j based on the joint life statuses of (x) and (y) is denoted by μ_{xy}^{ij} or $\mu_{x:y}^{ij}$, with the latter being preferred when we have numerical values for x and/or y.

Joint Life Model



The model above could be used in cases where cash flows depend on the status of two lives (x) and (y). If the model allows a transition from state 0 directly to state 3, it's known as a **common shock model**.

Traditional Multiple Life Status Actuarial Notation

The subscript xy denotes a status which fails upon the *first* of the future lifetimes of (x) and (y) to fail. Likewise, the status $\overline{xy}$ fails upon the failure of the *second* failure of (x) and (y).

- We will use these statuses in the same manner we used the single life status x and the term status $\overline{n|}$
- This notation follows the same convention we used earlier for term products and guaranteed annuities.

Then T_{xy} is the random variable describing the time until the *first* death of (x) and (y), and $T_{\overline{xy}}$ is the RV giving the time until the *second* death of (x) and (y).

Important Relationship

$$T_x + T_y = T_{xy} + T_{\overline{xy}}$$

Multiple Life Probability Definitions and Relationships

$$\mu_{xy} = \text{force of mortality for the joint status } xy$$
$$= \mu_{xy}^{01} + \mu_{xy}^{02} + \mu_{xy}^{03}$$

$$_tp_{xy} = P[\text{the status } xy \text{ does not fail within } t \text{ years}]$$
$$= P[(x) \text{ and } (y) \text{ are both alive in } t \text{ years}]$$
$$= {}_tp_{xy}^{00}$$

$$_tq_{xy} = P[\text{the status } xy \text{ fails within } t \text{ years}]$$
$$= 1 - {}_tp_{xy}$$
$$= P[(x) \text{ and } (y) \text{ are } \underline{\text{not}} \text{ both alive in } t \text{ years}]$$
$$= {}_tp_{xy}^{01} + {}_tp_{xy}^{02} + {}_tp_{xy}^{03}$$

Multiple Life Probability Definitions and Relationships

$$\begin{aligned}
{}_t p_{\overline{xy}} &= P[\text{the status } \overline{xy} \text{ does not fail within } t \text{ years}] \\
&= P[(x) \text{ or } (y) \text{ or both are still alive in } t \text{ years}] \\
&= {}_t p_{xy}^{00} + {}_t p_{xy}^{01} + {}_t p_{xy}^{02}
\end{aligned}$$

$$\begin{aligned}
{}_t q_{\overline{xy}} &= P[\text{the status } \overline{xy} \text{ fails within } t \text{ years}] \\
&= 1 - {}_t p_{\overline{xy}} \\
&= P[(x) \text{ and } (y) \text{ are both dead in } t \text{ years}] \\
&= {}_t p_{xy}^{03}
\end{aligned}$$

We have the following relationship for (x) and (y):

$$ {}_t p_x + {}_t p_y = {}_t p_{xy} + {}_t p_{\overline{xy}} $$

Further, we define ${}_n E_{xy} = v^n \, {}_n p_{xy}$ and ${}_n E_{\overline{xy}} = v^n \, {}_n p_{\overline{xy}}$

Multiple Life Example

In the joint life model, we have the following forces of transition:

$$\mu_{xy}^{01} = 0.03 + 0.0001xy \qquad \mu_{xy}^{02} = 0.02 + 0.001x + 0.002y$$

$$\mu_{xy}^{03} = 0.01 \qquad \mu_{x}^{13} = 0.03 + 0.002x + 0.0003x^2 \qquad \mu_{y}^{23} = 0.02$$

(a) Give an expression for μ_{xy}

(b) Calculate ${}_{10}p_{40:50}$ [0.00972]

(c) Give an expression for ${}_{5}p_{\overline{55:60}}$

More Multiple Life Probability Definitions and Relationships

We also use the numbers 1, 2, ... above the individual life statuses to indicate a specific order of status failure:

$$
\begin{aligned}
{}_tq_{\overset{1}{x}y} &= P[(x) \text{ dies before } (y) \text{ and within } t \text{ years}] \\
&= \int_0^t {}_sp_{xy}^{00}\, \mu_{x+s:y+s}^{02}\, ds
\end{aligned}
$$

$$
\begin{aligned}
{}_tq_{\overset{2}{x}y} &= P[(x) \text{ dies after } (y) \text{ and within } t \text{ years}] \\
&= \int_0^t {}_sp_{xy}^{01}\, \mu_{x+s}^{13}\, ds
\end{aligned}
$$

Note that we can let t be ∞ in either of these symbols.

We can derive some relationships among the various multiple life insurances and annuities:

$$\bar{A}_{xy} + \bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y \qquad \bar{a}_{xy} + \bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y$$

$$\bar{a}_{\overline{xy}} = \frac{1 - \bar{A}_{\overline{xy}}}{\delta} \qquad \bar{a}_{xy} = \frac{1 - \bar{A}_{xy}}{\delta}$$

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} \qquad \bar{a}_{x|y} + \bar{a}_{y|x} = \bar{a}_{\overline{xy}} - \bar{a}_{xy}$$

Two Independent Lives

In the special case where we're dealing with two future lifetimes that are independent, many of our expressions become much simpler. In this event, all of the forces of transition only depend on a single life, so in our joint life model, we would have:

$$\mu_{xy}^{01} = \mu_{y}^{23} = \mu_{y} \qquad \mu_{xy}^{02} = \mu_{x}^{13} = \mu_{x} \qquad \mu_{xy}^{03} = 0$$

As a result, our transition probabilities become functions of the individual survival probabilities:

$${}_tp_{xy}^{00} = {}_tp_x\,{}_tp_y \quad {}_tp_{xy}^{01} = {}_tp_x\,{}_tq_y \quad {}_tp_{xy}^{02} = {}_tq_x\,{}_tp_y \quad {}_tp_{xy}^{03} = {}_tq_x\,{}_tq_y$$

And for the case of independent lives, in the traditional actuarial notation, we have:

$$\mu_{xy} = \mu_x + \mu_y$$

$${}_tq_{\overline{xy}} = {}_tq_x\,{}_tq_y \quad \text{and} \quad {}_tp_{xy} = {}_tp_x\,{}_tp_y$$

Two Independent Lives – Example

Suppose that (x) and (y) have independent future lifetimes with $\mu_x = 0.02, \mu_y = 0.03,$ and $\delta = 0.05$.

(a) Calculate $\bar{a}_{x|y}$.

$$\bar{a}_{x|y} = \int_0^\infty e^{-\delta t}\, {}_tp_{xy}^{02}\, dt = \int_0^\infty e^{-\delta t}\, {}_tq_x\, {}_tp_y\, dt = 2.50$$

(b) Calculate the probability that (x) dies before (y).

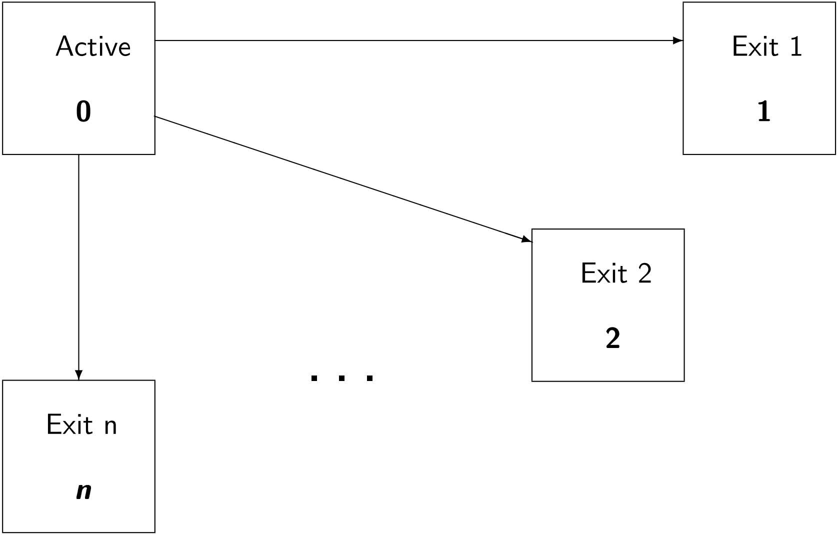
$$_\infty q_{\overset{1}{x}y} = \int_0^\infty {}_tp_{xy}^{00}\, \mu_{x+t:y+t}^{02}\, dt = \int_0^\infty {}_tp_x\, {}_tp_y\, \mu_{x+t}\, dt = 0.4$$

Multiple Decrement Models

Another common application of multi-state model theory is in situations where a members of a population may leave due to one of several different causes. These so-called **multiple decrement models** can applied to various situations.

- These models typically have one active / alive state, where each individual begins. An individual may leave the population by transitioning to any of the exit states.
- We usually only model a single transition out of the active state; we either ignore or disallow transitions among the exit states.

Multiple Decrement Model



Multiple Decrement Example

We issue a fully continuous whole life policy to (x). The death benefit is \$100,000, but doubles if (x) dies as a result of an accident. No benefit or surrender value is payable if the policyholder lapses his policy.

(a) Draw an appopriate state diagram for modeling this type of policy.

(b) Write an expression for the probability that this policy ever pays a benefit.

(c) Write an expression for the EPV of this insurance.

Multiple Decrement Tables and Traditional Actuarial Notation

The probabilities for multiple decrement models are traditionally expressed in **multiple decrement tables**, which are analogous to the mortality tables used in our alive-dead model.

- We again start with some cohort of ℓ_{x_0} people at age x_0. The number of people remaining in the population at age x is given by ℓ_x.
- The number of people leaving at age x due to decrement i is denoted by $d_x^{(i)}$; the total number leaving during this year is denoted by $d_x^{(\tau)}$, where

$$d_x^{(\tau)} = \sum_i d_x^{(i)}$$

- Then $\ell_{x+1} = \ell_x - d_x^{(1)} - d_x^{(2)} - \cdots - d_x^{(n)} = \ell_x - d_x^{(\tau)}$

Sample Double Decrement Table

Here's an except of a table with two decrements:
(e.g., decrement 1 = death; decrement 2 = retirement)

x	ℓ_x	$d_x^{(1)}$	$d_x^{(2)}$
60	1000	11	10
61	979	12	10
62	957	13	10
63	934	14	10
64	910	15	10
⋮	⋮	⋮	⋮

We can calculate that of the original 1000 members of this population, $d_{62}^{(2)} = 10$ of them leave due to retirement in the third year, and $d_{63}^{(\tau)} = d_{63}^{(1)} + d_{63}^{(2)} = 24$ of them leave in the fourth year for any reason.

Multiple Decrement Tables — More Notation

We denote the probability of leaving in a given year as a result of decrement i by

$$q_x^{(i)} = \frac{d_x^{(i)}}{\ell_x}$$

and the probability of leaving due to any decrement as

$$q_x^{(\tau)} = \sum_i q_x^{(i)} = \frac{d_x^{(\tau)}}{\ell_x}$$

We denote the probability of surviving **all** decrements in a given year by

$$p_x^{(\tau)} = 1 - q_x^{(\tau)} = \frac{\ell_{x+1}}{\ell_x}$$

Multiple Decrements in Continuous Models

In a continuous model, the force of failure for decrement i at age x is denoted by $\mu_x^{(i)}$ and the total force of failure is

$$\mu_x^{(\tau)} = \mu_x^{(1)} + \mu_x^{(2)} + \cdots + \mu_x^{(n)}$$

The probability of failing within t years due to decrement i is:

$${}_t q_x^{(i)} = \int_0^t {}_s p_x^{(\tau)} \, \mu_{x+s}^{(i)} \, ds$$

and the probability of failing within t due to any decrement is:

$${}_t q_x^{(\tau)} = \int_0^t {}_s p_x^{(\tau)} \, \mu_{x+s}^{(\tau)} \, ds$$

(Note that these aren't actually new formulas; they're just expressed in different notation.)

Dependent and Independent Probabilities for Multiple Decrement Models

In our multiple decrement table above, the $d_x^{(i)}$ values (and hence the calculated $q_x^{(i)}$ values) depended on the other decrements in the table; hence, $q_x^{(i)}$ is sometimes called a **dependent probability of decrement**.

If we calculate the probability of someone age x failing due to decrement i in the *absence* of other decrements, the resulting probability is called the **independent probability of decrement**, the **probability of decrement in the associated single decrement table**, or sometimes the **absolute rate of decrement** and is denoted by $q_x'^{(i)}$.

Note that ${}_tq_x'^{(i)} \geq {}_tq_x^{(i)}$ for any $x, t,$ and i.

Formulas for Independent Probabilities in Multiple Decrement Models

The probability of (x) still being in the population in t years in the associated single decrement model for decrement i is denoted ${}_tp_x'^{(i)}$.

$$ {}_tp_x'^{(i)} = 1 - {}_tq_x'^{(i)} \qquad {}_tp_x^{(\tau)} = \prod_{i=1}^{n} {}_tp_x'^{(i)} $$

$$ {}_tp_x'^{(i)} = \exp\left[-\int_0^t \mu_{x+s}^{(i)}\, ds\right] \qquad {}_tp_x^{(\tau)} = \exp\left[-\int_0^t \mu_{x+s}^{(\tau)}\, ds\right] $$

$$ \mu_{x+t}^{(i)} = \frac{-\frac{d}{dt}\,{}_tp_x'^{(i)}}{{}_tp_x'^{(i)}} $$

If we have a discrete model (i.e., a multiple decrement table), we'll often need to make some assumptions regarding how the decrements act within the year:

- We need this in order to get the independent failure probabilities from the dependent ones (and vice-versa).
- We also need this additional information to do calculations for non-integral lengths of time.

Example: For the double decrement table above, calculate and interpret $q'^{(1)}_{62}$ and $q'^{(2)}_{62}$ assuming:

(a) All retirements occur at the beginning of the year.
(b) All retirements occur at the end of the year.

(Assume in all cases that deaths happen throughout the year.)

Uniform Distribution of Decrements in the Multiple-Decrement Table

One option is to assume that each decrement in the multiple decrement table is distributed uniformly over the year in the *presence* of other decrements, i.e., ${}_tq_x^{(j)}$ is a linear function of t.

Under this "UDDM" assumption:

$$ {}_tq_x^{(j)} = t \cdot q_x^{(j)} $$

From this assumption, we can derive the following relationships:

$$ {}_tq_x^{(\tau)} = t \cdot q_x^{(\tau)} \qquad {}_tp_x^{(\tau)} = 1 - t \cdot q_x^{(\tau)} \qquad \mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{1 - t \cdot q_x^{(\tau)}} $$

$$ {}_tp_x^{\prime(j)} = \left(1 - t \cdot q_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}} $$

Uniform Distribution in the Associated Single-Decrement Tables

Another option is to assume that each decrement is distributed uniformly in its associated single decrement table, i.e., in the *absence* of other decrements, so that ${}_tq_x'^{(j)}$ is a linear function of t.

Under this "UDDS" assumption:

$${}_tq_x'^{(j)} = t \cdot q_x'^{(j)}$$

From this assumption, we can derive various relationships. For example, for a double decrement table, we have:

$$q_x^{(1)} = q_x'^{(1)} \left(1 - \frac{1}{2} \cdot q_x'^{(2)}\right) \quad \text{and} \quad q_x^{(2)} = q_x'^{(2)} \left(1 - \frac{1}{2} \cdot q_x'^{(1)}\right)$$

Similar equations can be derived for triple decrement tables.

Constant Force of Decrement Assumption

We could instead assume that the forces of decrement are constant within the year. This assumption results in the following formulas:

$$ {}_t p_x'^{(j)} = \left[p_x'^{(j)} \right]^t \qquad {}_t p_x^{(\tau)} = \left[p_x^{(\tau)} \right]^t $$

$$ {}_t q_x^{(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}} \left(1 - \left[p_x^{(\tau)} \right]^t \right) $$

$$ p_x'^{(j)} = \left[p_x^{(\tau)} \right]^{q_x^{(j)}/q_x^{(\tau)}} $$